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HW4

MATH 527A
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1. $f : V \to W$ is a linear map where V and W are normed linear spaces. f is continuous if $x_n \to x$ in V implies $f(x_n) \to f(x)$ in W. Show that the following are equivalent.

   (a) f is continuous.

   (b) f is continuous at 0.

   (c) There is a constant $C < \infty$ such that $||f(x)||_W \leq C||x||_V$ for all $x \in V$.

   This result can be paraphrased as follows: A linear mapping is continuous iff it is bounded.

   Solution. $(a) \implies (b)$

   Proof. Suppose $f : V \to W$ is a continuous linear map where V and W are normed linear spaces. Because f is continuous if $x_n \to x$ in V then $f(x_n) \to f(x)$ in W. Thus any sequence x_n that converges to 0 we have $f(x_n) \to f(0)$, thus f is continuous at 0.

   $(b) \implies (c)$

   Proof. Suppose that f is continuous at 0. We now use the δ, ϵ definition of continuity. For $\epsilon = 1$, there exists $\delta > 0$ such that,

$$||f(v) - f(0)||_W = ||f(v)||_W < 1,$$

   for all $||v||_V < \delta$. Now for all $x \in V$ consider,

$$||f(x)||_W = \left|\left| f\left(\left(\frac{cx}{||x||_V}\right)\left(\frac{||x||_V}{c}\right)\right)\right|\right|_W,$$

   where we choose $c \leq \delta$. Next we use linearity of f to get,

$$||f(x)||_W = \frac{||x||_V}{c}\left|\left| f\left(\frac{cx}{||x||_V}\right)\right|\right|_W.$$

   Notice that $||cx/||x||_V||_V = c \leq \delta$. Thus we can use continuity at zero for $v = cx/||x||_V$. Thus it follows,

$$||f(x)||_W \leq 1\left|\left|\frac{x}{c}\right|\right|_V = \frac{1}{c}||x||_V.$$

   Let $C = 1/c$ and we have the inequality that for all $x \in V$,

$$||f(x)||_W \leq C||x||_V.$$

   $(c) \implies (a)$

Proof. Suppose that there is a constant $C < \infty$ such that

$$||f(x)||_W \leq C||x||_V \quad \text{for all } x \in V.$$

Let $x_n \to x$. Then for $\epsilon/C > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $||x_n - x||_V < \epsilon/C$. From the inequality assumed, we have for all $n \geq N$,

$$||f(x_n) - f(x)||_W < \epsilon \quad \text{for all } n \geq N.$$

So $x_n \to x$ in V implies that $f(x_n) \to f(x)$ in W. Therefore, f is continuous.

Therefore, (a), (b), and (c) are equivalent.

2. $||\cdot||_a$ and $||\cdot||_b$ are two norms on a vector space V.

Show that the two norms are equivalent iff the identity map from $(V, ||\cdot||_a)$ to $(V, ||\cdot||_b)$ and its inverse are both continuous.

Solution. ($\Longrightarrow$)

Assume $||\cdot||_a, ||\cdot||_b$ are equivalent on V. We need to show that ϕ, ϕ^{-1} are continuous, where ϕ is the identity map from $(V, ||\cdot||_a)$ to $(V, ||\cdot||_b$. For ϕ, the following is true

$$\begin{aligned} v_n &\to v \in (V, ||\cdot||_a) \implies \phi(v_n) = v_n \to v = \phi(v) \in (V, ||\cdot||_a) \equiv (V, ||\cdot||_b) \\ v_n &\to v \in (V, ||\cdot||_b) \implies \phi^{-1}(v_n) = v_n \to v = \phi^{-1}(v) \in (V, ||\cdot||_b) \equiv (V, ||\cdot||_a) \end{aligned}$$

where I have used the equivalence of norms to make the final argument in each of the lines above. Each line above demonstrates the definition of continuity for a linear map given in exercise 1.

$\therefore$ two norms equivalent $\implies$ the identity map is continuous from $(V, ||\cdot||_a)$ to $(V, ||\cdot||_b$.

($\Longleftarrow$)

Assume ϕ, ϕ^{-1} as previously defined are continuous. Using the result of exercise 1,

$$\forall v \in V, \ \exists C_1, C_2 < \infty \text{ s.t. } ||v||_b \leq C_1||v||_a \text{ and } ||v||_a \leq C_2||v||_b.$$

The statement above can be restated as follows

$$\forall v \in V, \ \exists C_1, C_2' < \infty \text{ s.t. } C_2'||v||_a \leq ||v||_b \leq C_1||v||_a$$

where $C_2' = \frac{1}{C_2}$. But, this statement was proven to be equivalent to the statement "$||\cdot||_a, ||\cdot||_b$ are equivalent" in homework 3, exercise 2a.

$\therefore$ the identity map and its inverse continuous $\implies$ the two norms are equivalent.

Therefore, two norms, $||\cdot||_a, ||\cdot||_b$, are equivalent iff the identity map from $(V, ||\cdot||_a)$ to $(V, ||\cdot||_b)$ and its inverse are both continuous. $\square$

3. Show that the mapping defined by

$$f(x) \mapsto Tf(x) = \int_0^{\pi} \sin(x-y) f(y) dy$$

maps functions in $C([0,\pi])$ to $C([0,\pi])$. Is this mapping T bounded (or equivalently continuous) with respect to the L^1 and L^∞ norms on $C([0,\pi])$, and if so, what are the corresponding induced norms for T? (Hint: Think about the equivalent problem for matrices)

Solution. It is easy to show that $Tf(x)$ maps $f \in C([0,\pi])$ to $C([0,\pi])$. Since the integral of a continuous function is itself a continuous function and the fact that our domain of interest is $[0,\pi]$, $Tf(x)$ maps functions in $C([0,\pi])$ to $C([0,\pi])$.

To show T is bounded wrt the L^1-norm, consider the hint. Much of the legwork for this argument about the equivalent problem for matrices was developed in class. Consider

$$\begin{aligned} ||Tf||_1 &= \int_0^{\pi} dy |f(y)| \left| \int_0^{\pi} \sin(x-y) dx \right| \\ &\leq ||f||_1 || \cos(\pi - y) - \cos(y)||_\infty \text{ by Hölder's inequality} \\ &= 2||f||_1. \end{aligned}$$

To show T is bounded wrt the L^∞-norm, consider

$$\begin{aligned} ||Tf||_\infty &= \sup_{x\in[0,\pi]} \left| \int_0^{\pi} \sin(x-y) f(y) dy \right| \\ &\leq \sup_{x\in[0,\pi]} \int_0^{\pi} |\sin(x-y)||f(y)| dy \\ &\leq \sup_{x\in[0,\pi]} ||\sin(x-y)||_1 ||f||_\infty \\ &= \sup_{x\in[0,\pi]} (2 - (\cos(x-\pi) + \cos(x)))||f||_\infty \\ &= 2||f||_\infty. \end{aligned}$$

Therefore, T is bounded wrt both the L^1 and L^∞ norms. To show that the induced norm is indeed 2, need to show equality. For the L^1 case, pick function f with L^1 norm equal to 1, say $f(x) = \frac{\pi+1}{\pi(x+1)^2}$. Not so clearly, but nevertheless true, $||f||_1 = 1$. This says $||T|| = 2$. For the L^∞ case, pick function f with L^1 norm equal to 1. $f(x) = 1$, the constant function works just fine.

$\therefore ||T|| = 2$.

4. **Poincaré inequality**

(a) Show that,

$$||f||_{2,1} = \sqrt{\int_0^1 ([f(t)]^2 + [f'(t)]^2) dt}$$

defines a norm on $C^1([0,1])$.

Solution. Need to show non-degeneracy, symmetry, and the triangle inequality.

To be non-degenerate, $||f||_{2,1} \geq 0$ and $||f||_{2,1} = 0 \implies f = 0$. Due to non-degeneracy of the L^2-norm on $C^1([0,1])$ it follows directly that $\sqrt{\int_0^1 ([f(t)]^2 + [f'(t)]^2)dt} \geq 0$. If $f = 0$, $||f||_{2,1} = 0$, but is it exclusive? Since the integrand of $||f||_{2,1}$ is non-negative definite, the integral is zero only when $f = f' = 0$. Therefore $||f||_{2,1}$ is non-degnerate.

Symmetry is easy to show. For $f, g \in C^1([0,1])$,

$$\begin{aligned} ||f+g||_{2,1} &= \sqrt{\int_0^1 ([f(t)+g(t)]^2 + [f'(t)+g'(t)]^2)dt} = \sqrt{\int_0^1 ([g(t)+f(t)]^2 + [g'(t)+f'(t)]^2)dt} \\ &= ||g+f||_{2,1}. \end{aligned}$$

To show the triangle inequality, define an inner product $< \cdot, \cdot >: C^1 \times C^1 \to \mathbb{R}$ by

$$< f, g >= \int_0^1 [f(t)g(t) + f'(t)g'(t)]dt.$$

Then a direct computation yields $< f, f >= ||f||_{2,1}^2 \geq 0$. By nondegeneracy of $|| \cdots ||_{2,1}$, we see that, for all real λ and any $f, g \in C^1$ we have

$$< f - \lambda g, f - \lambda g >= ||f||_{2,1}^2 - 2\lambda < f, g > + \lambda^2 ||g||_{2,1}^2 \geq 0$$

This implies that the quadratic form (in λ) cannot have distinct real roots, implying

$$| < f, g > | \leq ||f||_{2,1} ||g||_{2,1}$$

which is the analog of Hölder's inequality for this norm. Now, we have

$$\begin{aligned} ||f+g||_{2,1}^2 = ||f||_{2,1}^2 + 2 < f, g > + ||g||_{2,1}^2 & \\ &\leq ||f||_{2,1}^2 + 2||f||_{2,1}||g||_{2,1} + ||g||_{2,1}^2 \\ &= (||f||_{2,1} + ||g||_{2,1})^2 \end{aligned}$$

Taking the square root yields the triangle inequality.

(b) If $f \in C^1([0,1])$, show that, there exists a constant C such that

$$||f||_\infty = \sup_{t\in[0,1]} |f(t)| \leq C||f||_{2,1}$$

Solution. Suppose that for some function g that $\exists c \in (0,1)$ such that $g(c) = 0$. It follows that

$$|g(x)| = |g(x) - g(c)| = |\int_x^c g'(t)dt| \leq \int_x^c |g'(t)|dt$$

Since $c \in (0,1)$, $\forall x \in [0,1]$, $g(x) \leq \int_0^1 |g'(t)|dt$. A nice feature is that there is no dependence on x, so it follows that $\|g\|_\infty = \max_{x\in[0,1]} |g(x)| \leq \int_0^1 |g'(t)|dt$.

Consider g defined by $g(x) = f(x) - \bar{f}(x)$, where $\bar{f}(x) = \int_0^1 f(t)dt$ is the average value of f.

Now apply the Mean Value Theorem, which states in essence that since $\bar{f}$ assumes the average value of f, $\exists c$ such that $f(c) = \int_0^1 f(t)dt \Rightarrow g(c) = 0$. Now consider the ∞-norm of f, which can be written as

$$\|f\|_\infty = \|g + \bar{f}\|_\infty \leq \|g\|_\infty + |\bar{f}|$$

since $\bar{f}$ is constant.

We can now use the inequality derived earlier for $||g||_\infty$ to get

$$\|g\|_\infty + |\bar{f}| \leq \int_0^1 |g'(t)|dt + |\int_0^1 (f)dt| \leq \int_0^1 |f'(t)|dt + \int_0^1 |f|dt$$

Which we find since (1) $g(t) = f(t) - \bar{f}(t)$, (2) $g'(t) = f'(t)$, and (3) $|\int_0^1 (f)dt| \leq \int_0^1 |f|dt$. Now we have $\int_0^1 |f'(t)|dt + \int_0^1 |f|dt \leq \int_0^1 |f'(t) \cdot 1|dt + \int_0^1 |f \cdot 1|dt$ to which we can apply Hölder's inequality two times

$$\int_0^1 |f'(t) \cdot 1|dt + \int_0^1 |f \cdot 1|dt \leq \sqrt{\int_0^1 1^2 dt} \left(\sqrt{\int_0^1 |f'|^2 dt} + \sqrt{\int_0^1 |f|^2 dt} \right)$$

$$= \sqrt{\int_0^1 |f'|^2 dt} + \sqrt{\int_0^1 |f|^2 dt}$$

$$\sqrt{\int_0^1 |f'|^2 dt} + \sqrt{\int_0^1 |f|^2 dt} \leq \sqrt{\int_0^1 (|f'|^2 + |f|^2) dt} + \sqrt{\int_0^1 (|f|^2 + |f'|^2) dt}$$

Which we have since we are adding two non-negative terms $|f|^2$ and $|f'|^2$. It follows from above then that

$$\|f\|_\infty \leq \|f\|_{2,1} + \|f\|_{2,1} = 2\|f\|_{2,1}$$

$\therefore$ There exists a constant, $C = 2$, s.t.

$$||f||_\infty = \sup_{t \in [0,1]} |f(t)| \leq C||f||_{2,1}$$